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Macroscopic modelling of heat transfer in composites with interfacial thermal barrier

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Abstract—Macroscopic modelling of heat transfer in composites in the presence of interfacial thermal resistance is determined from a description at the heterogeneity scale using a method of double-scale asymptotic developments. The analysis shows five different models. Their domains of validity are defined by the value of a dimensionless parameter. That permits the determination of the correct model for a given macroscopic boundary value problem. The models for a binary composite are shown to belong to two main types: one-temperature and two-temperature field models.

1. INTRODUCTION

Analytical studies have shown that the effective thermal conductivity of a composite depends on the value of the thermal conductivity, the volume fraction and the distribution of the individual components within the composite. Recent investigations have also identified the critical role of the interfacial thermal barrier resistance in the effective thermal conductivity of composites [1–8]. In these studies the model to describe the effective behaviour is assumed to be a one-temperature field model, and the authors look for the effective conductivity and its relation to the interfacial barrier resistance. Investigations are either theoretical [1–4, 6, 8] or experimental [5, 7], for dilute volume fractions [2], low concentrations [3] or high concentrations [1, 4–8]. The thermal barrier is shown to lower the effective thermal conductivity of the composite. On an other hand two-temperature field models are introduced to describe porous matrix–liquid systems [9–12] or heterogeneous solids [13, 14]. Thermal barrier resistances are not explicitly introduced but the interphase heat transfer is equivalently assumed to be proportional to the temperature difference.

The aim of the present paper is to determine the influence of the interfacial thermal barrier on the effective thermal conductivity and on the structure of the macroscopic heat transfer equations. Since periodic and random micro structures lead to similar macroscopic behaviour [15], we assume without loss of generality that the composite is periodic. The method of asymptotic developments will be used for homogenization.

Macroscopic heat transfer in periodic composites using the method of asymptotic developments [16, 17] was studied in ref. [18] with the classical boundary conditions between the constituents, e.g. continuity of temperature and normal flux. The coupled problem of the thermoelasticity of composites was investigated in ref. [19].

In Section 2, the bases of the method of asymptotic developments are recalled, as is the description of transient heat transfer at the micro level. We assume a two-component composite with conductivities of the same order of magnitude. Sections 2–6 investigate five characteristic cases related to different relative values of the barrier resistance to the resistance of the components. Finally, the five corresponding macroscopic models are illustrated in Section 7, where a layered medium is investigated which permits analytical results. The first three models are one-temperature field models whereas the last two are two-temperature field models. Relations among the five models are analyzed in Section 8. We show that heat transfer in a given composite is not *a priori* described by a single model. The choice of model also depends on the excitation.

2. FORMULATION OF THE PROBLEM: HOMOGENIZATION

For simplicity we assume that the medium is composed of two solids. The results can be easily extended to n arbitrary solids. We consider a composite with a fine periodic structure. A period Y is shown in Fig. 1. Solids 1 and 2 occupy the domains Y_1 and Y_2, respectively. Γ denotes their common boundary. For

NOMENCLATURE

A, B	dimensionless numbers
$a^{\varepsilon}_{ij}, a_{ij}$	conductivity of medium 1
$b^{\varepsilon}_{ij}, b_{ij}$	conductivity of medium 2
$\tilde{C}$	effective heat capacity of the composite
$C^{\varepsilon}_1, C^{\varepsilon}_2$	heat capacities of media 1 and 2, respectively
H	effective heat transfer coefficient of composite
h^{ε}, h	interfacial thermal conductance
L	characteristic macroscopic length
l	characteristic microscopic length
n	partial volume of constituent 2
n_i	unit normal to Γ
Q	Biot number
T_i	temperature field in medium i
t	time
X_i	dimensional space variable
x_i	dimensionless slow space variable
Y	spatial period
Y_i	domain occupied by medium i in period Y
y_i	dimensionless fast space variable.

Greek symbols

δ_{ij}	identity matrix
ε	small parameter of separation of scales
Γ	common boundary of the two media
$\lambda^{\alpha\,\mathrm{eff}}_{ij}$	bulk effective conductivity of composite in case α
$\lambda^{\alpha\,\mathrm{eff}}_{1ij}, \lambda^{\alpha\,\mathrm{eff}}_{2ij}$	effective conductivities of media 1 and 2 in case α
τ	characteristic time
χ^{α}_i	particular solution for T.

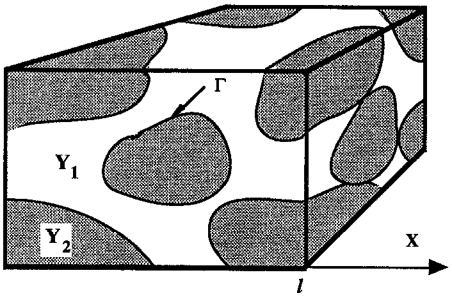

Fig. 1. Periodic cell of two-constituent composite.

simplicity we also assume that each domain Y_i, $i = 1, 2$, is connected. The case where one or both of the two components are not connected can be easily investigated by following the same procedure. The reader is referred to Section 7, where an example is shown.

The period Y, of dimension $O(l)$, is small compared to the characteristic length L of the medium submitted to heat transfer at the macroscopic scale:

$$\varepsilon = \frac{l}{L} \ll 1. \quad (2.1)$$

For a steady heat transfer, L can be assimilated to the characteristic size of the macroscopic sample. For unsteady heat transfer, e.g. at constant pulsation, a good candidate for L is $\lambda/2\pi$, where λ is the wavelength.

The set of equations giving the temperature at the micro-scale is:

$$C^{\varepsilon}_1 \frac{\partial T^{\varepsilon}_1}{\partial t} = \frac{\partial}{\partial X_i}\left(a^{\varepsilon}_{ij}\frac{\partial T^{\varepsilon}_1}{\partial X_j}\right) \quad \text{in } Y_1 \quad (2.2)$$

$$C^{\varepsilon}_2 \frac{\partial T^{\varepsilon}_2}{\partial t} = \frac{\partial}{\partial X_i}\left(b^{\varepsilon}_{ij}\frac{\partial T^{\varepsilon}_2}{\partial X_j}\right) \quad \text{in } Y_2 \quad (2.3)$$

where a^{ε}_{ij} and b^{ε}_{ij} are the thermal conductivities, C^{ε}_1 and C^{ε}_2 the volume heat capacities, and T^{ε}_1 and T^{ε}_2 the temperature in solids 1 and 2, respectively. The superscript ε recalls the finely heterogeneous character of the micro-scale.

The boundary and initial conditions [9] are as follows:

$$a^{\varepsilon}_{ij}\frac{\partial T^{\varepsilon}_1}{\partial X_j} n_i = b^{\varepsilon}_{ij}\frac{\partial T^{\varepsilon}_2}{\partial X_j} n_i \quad \text{on } \Gamma \quad (2.4)$$

$$-a^{\varepsilon}_{ij}\frac{\partial T^{\varepsilon}_1}{\partial X_j} n_{1i} = h^{\varepsilon}(T^{\varepsilon}_1 - T^{\varepsilon}_2) \quad \text{on } \Gamma \quad (2.5)$$

$$T^{\varepsilon}_1 = 0 \quad T^{\varepsilon}_2 = 0 \quad \text{for } t = 0 \quad (2.6)$$

where n_1 is the outward normal to Y_1, and $h^{\varepsilon} > 0$ is the interfacial thermal conductance. The conductivities, heat capacities and interfacial conductance are Y-periodic, and there exists $\alpha > 0$, with $a^{\varepsilon}_{ij}\xi_i\xi_j \geqslant \alpha\xi_i\xi_i$, $\forall\xi \in R^3$, and $\beta > 0$ with $b^{\varepsilon}_{ij}\xi_i\xi_j \geqslant \beta\xi_i\xi_i$, $\forall\xi \in R^3$.

Set (2.4)–(2.8) introduces three dimensionless numbers:

$$A = \frac{|a|}{|b|} \quad B_l = \frac{\left|C\frac{\partial T}{\partial t}\right|}{\left|\frac{\partial}{\partial X_i}\left(a^{\varepsilon}_{ij}\frac{\partial T}{\partial X_j}\right)\right|} = \frac{Cl^2}{|a|\tau}$$

$$Q_l = \frac{|h^{\varepsilon}(T_1 - T_2)|}{\left|a^{\varepsilon}_{ij}\frac{\partial T}{\partial X_j} n_i\right|} = \frac{hl}{|a|} \quad (2.7)$$

where the index l shows that the characteristic length l has been used to define these dimensionless numbers, and τ is a characteristic time for the heat transfer, e.g. $\tau = 2\pi/\omega$ for constant-pulsation heat transfer. $|a|$ stands for an estimation of the conductivity a. It is

Fig. 2. Different macroscopic models with respect to Q_l.

obvious that an equivalent continuous macroscopic description depends on the value of these numbers. As already mentioned, we assume that the conductivities of the two components are of the same order of magnitude. Therefore $A = O(1)$. On the other hand, a transient heat transfer is homogenizable, i.e. with an equivalent macroscopic description, only if (see ref. [20], where a similar problem is investigated concerning diffusion)

$$B_l = O(\varepsilon^2). \tag{2.8}$$

This condition shows that, for a given material, the characteristic time τ must be sufficiently large to preserve a good scale separation. For example, considering a harmonic excitation, decreasing τ will decrease the wavelength, i.e. L. In the limit, $L = O(l)$ and there is no more scale separation. An equivalent macroscopic description does not exist.

Finally the Biot number Q_l is a measure of the conductivity of the interfacial barrier with respect to the conductivity of the components. The cases of interest are

$$Q_l = O(\varepsilon^p) \quad p = -1, 0, 1, 2, 3$$

corresponding to models I, II, III, IV and V, respectively (Fig. 2). Hereafter the superscripts I–V refer to these models. With the above estimations of the dimensionless numbers and the dimensionless space variable $y = X/l$, the dimensionless micro-description is:

$$\varepsilon^2 C_1^\varepsilon \frac{\partial T_1^\varepsilon}{\partial t} = \frac{\partial}{\partial y_i}\left(a_{ij}^\varepsilon \frac{\partial T_1^\varepsilon}{\partial y_j}\right) \quad \text{in } Y_1 \tag{2.9}$$

$$\varepsilon^2 C_2^\varepsilon \frac{\partial T_2^\varepsilon}{\partial t} = \frac{\partial}{\partial y_i}\left(b_{ij}^\varepsilon \frac{\partial T_2^\varepsilon}{\partial y_j}\right) \quad \text{in } Y_2 \tag{2.10}$$

$$a_{ij}^\varepsilon \frac{\partial T_1^\varepsilon}{\partial y_j} n_i = b_{ij}^\varepsilon \frac{\partial T_2^\varepsilon}{\partial y_j} n_i \quad \text{on } \Gamma \tag{2.11}$$

$$-a_{ij}^\varepsilon \frac{\partial T_1^\varepsilon}{\partial y_j} n_{1i} = \varepsilon^p h^\varepsilon (T_1^\varepsilon - T_2^\varepsilon) \quad \text{on } \Gamma \tag{2.12}$$

$$T_1^\varepsilon = 0 \quad T_2^\varepsilon = 0 \quad \text{for } t = 0 \tag{2.13}$$

where, for the sake of simplicity, the notation is left unchanged.

The macroscopic behaviour of the heat transfer is measured using the homogenization method [15–17] based on an asymptotic expansion of the power of the small parameter ε and including a double scale with characteristic lengths l and L. These characteristic lengths introduce two dimensionless space variables, $y = X/l$ and $x = X/L$. The variable y is the microscopic one, describing the small heterogeneities, while x is the macroscopic space variable. The temperature T^ε is a function of the two space variables, x and y:

$$T^\varepsilon = T(x,y) \quad x = \varepsilon y.$$

For simplicity we assume that the composite is strictly periodic:

$$a_{ij}^\varepsilon = a_{ij}(y) \quad b_{ij}^\varepsilon = b_{ij}(y) \quad C_1^\varepsilon = C_1(y)$$
$$C_2^\varepsilon = C_2(y) \quad h^\varepsilon = h(y).$$

Then we search for an asymptotic expansion for T^ε of the form

$$T_i(x,y) = T_i^0(x,y) + \varepsilon T_i^1(x,y) + \dots$$
$$x = \varepsilon y \quad i = 1,2 \tag{2.14}$$

where $T_i^k(x,y)$ are Y-periodic functions in y.

The method consists in incorporating expansion (2.14) into the dimensionless set, identifying the similar powers in ε and solving a set of boundary value problems in a characteristic cell Y. In the computation we must take into account the fact that x and y should be considered as independent variables and that the derivation operator is now expressed by

$$\frac{\partial T^\varepsilon}{\partial y_i} = \frac{\partial T}{\partial y_i} + \varepsilon \frac{\partial T}{\partial x_i}.$$

The homogenization process, $\varepsilon \to 0$, produces a set of equations satisfied by T^0, which in fact represent the macroscopic behaviour of the heat transfer in our composite. This formal computation was described completely from the mathematical point of view by the so-called "two-scale convergence". For details see, for example, refs. [21, 22].

For the first order of approximation, model I with $Q_l = O(\varepsilon^{-1})$ corresponds to a classical composite without barrier resistance. It is easy to check that the successive sets of boundary value problems to be solved are similar to those in ref. [18, Section 3], with an added transient term as in ref. [20]. Therefore the macroscopic description is

$$\tilde{C}\frac{\partial T^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{Ieff}} \frac{\partial T^0}{\partial x_i}\right). \tag{2.15}$$

The superscript I refers to model I. A tilde shows a volume average:

$$\tilde{\cdot} = \frac{1}{|Y|}\int_Y \cdot \, \mathrm{d}y.$$

The effective conductivity $\lambda_{ij}^{\mathrm{Ieff}}$ is also a volume average given by

$$\lambda_{ik}^{\text{Ieff}} = \frac{1}{|Y|}\int_Y \alpha_{ij}\left(\delta_{jk} + \frac{\partial \chi_k^{\text{I}}}{\partial y_j}\right)\text{d}y$$

$$\alpha = a \text{ in } Y_1 \text{ and } b \text{ in } Y_2$$

where the vector χ^{I} is Y-periodic and verifies the boundary value problem

$$\frac{\partial}{\partial y_i}\left[\alpha_{ij}\left(\delta_{jk} + \frac{\partial \chi_k^{\text{I}}}{\partial y_j}\right)\right] = 0 \quad \text{in } Y \quad \tilde{\chi}_k^{\text{I}} = 0$$

$$a_{ij}\left(\delta_{jk} + \frac{\partial \chi_k^{\text{I}}}{\partial y_j}\right)n_k = b_{ij}\left(\delta_{jk} + \frac{\partial \chi_k^{\text{I}}}{\partial y_j}\right)n_k \quad \text{on } \Gamma.$$

It is easy to show that the cases $Q_l = \text{O}(\varepsilon^p)$, $p < -1$, are described by model I.

3. MODEL II: $Q_l = \text{O}(1)$

Starting from the classical case I, we increase the barrier resistance to $Q_l = \text{O}(1)$. Expansion (2.14) will now be introduced into equations (2.9)–(2.12).

These equations at order ε^0 give:

$$\frac{\partial}{\partial y_i}\left[a_{ij}(y)\frac{\partial T_1^0}{\partial y_j}\right] = 0 \quad y \in Y_1 \tag{3.1}$$

$$\frac{\partial}{\partial y_i}\left[b_{ij}(y)\frac{\partial T_2^0}{\partial y_j}\right] = 0 \quad y \in Y_2 \tag{3.2}$$

$$a_{ij}(y)\frac{\partial T_1^0}{\partial y_j}n_i = b_{ij}(y)\frac{\partial T_2^0}{\partial y_j}n_i \quad y \in \Gamma \tag{3.3}$$

$$-a_{ij}(y)\frac{\partial T_1^0}{\partial y_j}n_{1i} = h(T_1^0 - T_2^0) \quad y \in \Gamma \tag{3.4}$$

$$T_1^0 \text{ and } T_2^0 \text{ } Y\text{-periodic.} \tag{3.5}$$

It is easy to observe that the only periodic solution of the problem is $T_1^0 = T_1^0(t,x)$, $T_2^0 = T_2^0(t,x)$. Then, with equation (3.4), $T_1^0(t,x) = T_2^0(t,x) = T^0(t,x)$.

Equations (2.9)–(2.12) at order ε give:

$$\frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_1^1}{\partial y_j}\right)\right] = 0 \quad y \in Y_1 \tag{3.6}$$

$$\frac{\partial}{\partial y_i}\left[b_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_2^1}{\partial y_j}\right)\right] = 0 \quad y \in Y_2 \tag{3.7}$$

$$a_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_1^1}{\partial y_j}\right)n_i = b_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_2^1}{\partial y_j}\right)n_i \quad y \in \Gamma \tag{3.8}$$

$$-a_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_1^1}{\partial y_j}\right)n_{1i} = h(T_1^1 - T_2^1) \quad y \in \Gamma \tag{3.9}$$

$$T_1^1 \text{ and } T_2^1 \text{ } Y\text{-periodic.} \tag{3.10}$$

Set (3.6)–(3.10) represents a linear differential problem with respect to the variable y. The unknowns T_1^1 and T_2^1 are linear functions of the gradient, $\partial T^0/\partial x_j$, to an arbitrary additive y independent function:

$$T^1 = \chi_i^{\text{II}}\frac{\partial T^0}{\partial x_i} + \bar{T}^1(t,x) \tag{3.11}$$

where T^1 stands for T_1^1 and T_2^1 in Y_1 and Y_2, respectively. The superscript II refers to model II. The functions χ_i^{II} are h-dependent. They are given by the following cell problem, where χ_{1k}^{II} and χ_{2k}^{II} stand for χ_k^{II} in Y_1 and Y_2, respectively:

$$-\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial \chi_{1k}^{\text{II}}}{\partial y_j}\right) = \frac{\partial a_{ik}}{\partial y_i} \quad y \in Y_1 \tag{3.12}$$

$$-\frac{\partial}{\partial y_i}\left(b_{ij}\frac{\partial \chi_{2k}^{\text{II}}}{\partial y_j}\right) = \frac{\partial b_{ik}}{\partial y_i} \quad y \in Y_2 \tag{3.13}$$

$$a_{ij}\frac{\partial \chi_{1k}^{\text{II}}}{\partial y_j}n_i = b_{ij}\frac{\partial \chi_{2k}^{\text{II}}}{\partial y_j}n_i \quad \text{on } \Gamma \tag{3.14}$$

$$a_{ij}\frac{\partial \chi_{1k}^{\text{II}}}{\partial y_j}n_{1i} + h(\chi_{1k}^{\text{II}} - \chi_{2k}^{\text{II}}) = -a_{ik}n_{1i} \quad \text{on } \Gamma \tag{3.15}$$

where χ_{1k}^{II} and χ_{2k}^{II} are Y-periodic.

The last step in the computation of the macroscopic behaviour is now to consider equations (2.9)–(2.11) order ε^2:

$$\frac{\partial}{\partial x_i}\left[a_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_1^1}{\partial y_j}\right)\right] + \frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\frac{\partial T_1^1}{\partial x_j} + \frac{\partial T_1^2}{\partial y_j}\right)\right] = C_1\frac{\partial T^0}{\partial t} \quad y \in Y_1$$

$$\frac{\partial}{\partial x_i}\left[b_{ij}(y)\left(\frac{\partial T^0}{\partial x_j} + \frac{\partial T_2^1}{\partial y_j}\right)\right] + \frac{\partial}{\partial y_i}\left[b_{ij}(y)\left(\frac{\partial T_2^1}{\partial x_j} + \frac{\partial T_2^2}{\partial y_j}\right)\right] = C_2\frac{\partial T^0}{\partial t} \quad y \in Y_2$$

$$a_{ij}(y)\left(\frac{\partial T_1^1}{\partial x_j} + \frac{\partial T_1^2}{\partial y_j}\right)n_i = b_{ij}(y)\left(\frac{\partial T_2^1}{\partial x_j} + \frac{\partial T_2^2}{\partial y_j}\right)n_i \quad y \in \Gamma.$$

Integrating the first two equations on Y_1 and Y_2, respectively, using the divergence theorem and the last equation, gives the following macroscopic behaviour:

$$\tilde{C}\frac{\partial T^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\text{IIeff}}\frac{\partial T^0}{\partial x_i}\right) \tag{3.16}$$

where the effective conductivity $\lambda_{ik}^{\text{IIeff}}$ is obtained from

$$\lambda_{ik}^{\text{IIeff}} = \frac{1}{|Y|}\int_Y \alpha_{ij}\left(\delta_{jk} + \frac{\partial \chi_k^{\text{II}}}{\partial y_j}\right)\text{d}y$$

$$\alpha = a \text{ in } Y_1 \text{ and } b \text{ in } Y_2.$$

It is important to remark that the effective thermal conductivity depends on h. That means that, at the macroscopic scale, we must consider not only the values of the thermal conductivities of the constituents, the volume fraction and the geometry of the composite, but also the interfacial thermal barrier resistance. Nevertheless, the structure of model II [equation (3.16)] remains classical, i.e. the heat transfer is described at the macroscopic scale by only a one-temperature field.

4. MODEL III: $Q_l = O(\varepsilon)$

Let us again increase the thermal barrier resistance. Now $p = 1$ and $Q_l = O(\varepsilon)$. The homogenization process, presented in detail in Section 3, is the same. For this reason we limit ourselves to the main points.

For order ε^0 equations (2.9)–(2.12) give

$$T_1^0 = T_1^0(t,x) \quad T_2^0 = T_2^0(t,x). \tag{4.1}$$

For order ε, we obtain the boundary value problem for T_1^1 and T_2^1, which are Y-periodic:

$$\frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\frac{\partial T_1^0}{\partial x_j}+\frac{\partial T_1^1}{\partial y_j}\right)\right] = 0 \quad y \in Y_1 \tag{4.2}$$

$$\frac{\partial}{\partial y_i}\left[b_{ij}(y)\left(\frac{\partial T_2^0}{\partial x_j}+\frac{\partial T_2^1}{\partial y_j}\right)\right] = 0 \quad y \in Y_2 \tag{4.3}$$

$$a_{ij}(y)\left(\frac{\partial T_1^0}{\partial x_j}+\frac{\partial T_1^1}{\partial y_j}\right)n_i = b_{ij}(y)\left(\frac{\partial T_2^0}{\partial x_j}+\frac{\partial T_2^1}{\partial y_j}\right)n_i \quad y \in \Gamma \tag{4.4}$$

$$-a_{ij}(y)\left(\frac{\partial T_1^0}{\partial x_j}+\frac{\partial T_1^1}{\partial y_j}\right)n_{1i} = h(T_1^0 - T_2^0) \quad y \in \Gamma. \tag{4.5}$$

We obtain two non-coupled problems for T_1^1 and T_2^1, i.e. equations (4.2), (4.5) and (4.3), (4.6), respectively, where equation (4.6) is derived from equations (4.4) and (4.5):

$$b_{ij}(y)\left(\frac{\partial T_2^0}{\partial x_j}+\frac{\partial T_2^1}{\partial y_j}\right)n_{2i} = h(T_1^0 - T_2^0) \quad y \in \Gamma. \tag{4.6}$$

Problem (4.2), (4.5) introduces a compatibility condition easily obtained by integrating equation (4.2) on Y_1, using the divergence theorem and the boundary condition (4.5). It yields

$$\int_\Gamma h(T_1^0 - T_2^0)\,\mathrm{d}s = (T_1^0 - T_2^0)\int_\Gamma h\,\mathrm{d}s = 0$$

and, since h is a positive quantity:

$$T_1^0(t,x) = T_2^0(t,x) = T^0(t,x). \tag{4.7}$$

Relation (4.7) also represents the compatibility condition of problem (4.3), (4.6).

The solutions of the two cell problems are therefore independent of h. They can be put in the form

$$T_1^1 = \chi_{1i}^{\mathrm{III}}\frac{\partial T^0}{\partial x_i} + \bar{T}_1^1(t,x) \quad T_2^1 = \chi_{2i}^{\mathrm{III}}\frac{\partial T^0}{\partial x_i} + \bar{T}_2^1(t,x). \tag{4.8}$$

The macroscopic behaviour is obtained for the following order as in Section 3. It becomes

$$\tilde{C}\frac{\partial T^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{ij}^{\mathrm{IIIeff}}\frac{\partial T^0}{\partial x_i}\right). \tag{4.9}$$

Now the effective conductivity is the sum of the effective conductivities of the two constituents, which are assumed to be perfectly insulated:

$$\lambda_{ij}^{\mathrm{IIIeff}} = \lambda_{1ij}^{\mathrm{IIIeff}} + \lambda_{2ij}^{\mathrm{IIIeff}}$$

$$\lambda_{1ik}^{\mathrm{IIIeff}} = \frac{1}{|Y|}\int_{Y_1} a_{ij}\left(\delta_{jk} + \frac{\partial \chi_{1k}^{\mathrm{III}}}{\partial y_j}\right)\mathrm{d}y$$

$$\lambda_{2ik}^{\mathrm{IIIeff}} = \frac{1}{|Y|}\int_{Y_2} b_{ij}\left(\delta_{jk} + \frac{\partial \chi_{2k}^{\mathrm{III}}}{\partial y_j}\right)\mathrm{d}y.$$

The macroscopic behaviour is classical, with a one-temperature field and an h-independent effective conductivity, but the thermal barrier resistance is sufficient to insulate the two constituents at the first order.

5. MODEL IV: $Q_l = O(\varepsilon^2)$

We again increase the thermal barrier resistance by a power of ε. As above the first order gives

$$T_1^0 = T_1^0(t,x) \quad T_2^0 = T_2^0(t,x). \tag{5.1}$$

Compared to the result in Section 4, the next order for T_1^1 and T_2^1 is given by equations (4.2)–(4.4) and relation (5.2) instead of equation (4.5):

$$-a_{ij}(y)\left(\frac{\partial T_1^0}{\partial x_j}+\frac{\partial T_1^1}{\partial y_j}\right)n_{1i} = 0 \quad y \in \Gamma. \tag{5.2}$$

The solutions therefore are similar to the solutions in Section 4, but with $T_1^0(t,x) \neq T_2^0(t,x)$:

$$T_1^1 = \chi_{1i}^{\mathrm{IV}}\frac{\partial T_1^0}{\partial x_i} + \bar{T}_1^1(t,x)$$

$$T_2^1 = \chi_{2i}^{\mathrm{IV}}\frac{\partial T_2^0}{\partial x_i} + \bar{T}_2^1(t,x) \tag{5.3}$$

with $\chi_{1i}^{\mathrm{IV}} = \chi_{1i}^{\mathrm{III}}$, $\chi_{2i}^{\mathrm{IV}} = \chi_{2i}^{\mathrm{III}}$.

For order ε^2 it becomes

$$\frac{\partial}{\partial x_i}\left[a_{ij}(y)\left(\frac{\partial T^0}{\partial x_j}+\frac{\partial T_1^1}{\partial y_j}\right)\right] + \frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\frac{\partial T_1^1}{\partial x_j}+\frac{\partial T_1^2}{\partial y_j}\right)\right] = C_1\frac{\partial T^0}{\partial t} \quad y \in Y_1$$

$$a_{ij}(y)\left(\frac{\partial T_1^1}{\partial x_j}+\frac{\partial T_1^2}{\partial y_j}\right)n_{1i} = -h(T_1^0-T_2^0) \quad y\in\Gamma$$

$$\frac{\partial}{\partial x_i}\left[b_{ij}(y)\left(\frac{\partial T^0}{\partial x_j}+\frac{\partial T_2^1}{\partial y_j}\right)\right] + \frac{\partial}{\partial y_i}\left[b_{ij}(y)\left(\frac{\partial T_2^1}{\partial x_j}+\frac{\partial T_2^2}{\partial y_j}\right)\right] = C_2\frac{\partial T^0}{\partial t} \quad y\in Y_2$$

$$b_{ij}(y)\left(\frac{\partial T_2^1}{\partial x_j}+\frac{\partial T_2^2}{\partial y_j}\right)n_{2i} = h(T_1^0-T_2^0) \quad y\in\Gamma.$$

By integrating the first and third equations on Y_1 and Y_2, respectively, one obtains two coupled macroscopic equations for the two temperature fields, T_1^0 and T_2^0:

$$\tilde{C}_1\frac{\partial T_1^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{1ij}^{\mathrm{IVeff}}\frac{\partial T_1^0}{\partial x_i}\right) - H(T_1^0-T_2^0) \quad (5.4)$$

$$\tilde{C}_2\frac{\partial T_2^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{2ij}^{\mathrm{IVeff}}\frac{\partial T_2^0}{\partial x_i}\right) + H(T_1^0-T_2^0) \quad (5.5)$$

with $\lambda_{1ij}^{\mathrm{IVeff}} = \lambda_{1ij}^{\mathrm{IIIeff}}$, $\lambda_{2ij}^{\mathrm{IVeff}} = \lambda_{2ij}^{\mathrm{IIIeff}}$ and

$$H = \frac{1}{|Y|}\int_\Gamma h\,\mathrm{d}s. \quad (5.6)$$

The macroscopic behaviour is now a two-temperature field description. As in Section 4 the effective conductivity of each constituent is obtained by considering it as being insulated.

6. MODEL V: $Q_l = O(\varepsilon^3)$

Finally we investigate the case where the thermal barrier resistance is very large, $Q_l = O(\varepsilon^3)$. By following the same procedure again it is obvious that the macroscopic behaviour is given by equations (5.4) and (5.5), but without the source terms:

$$\tilde{C}_1\frac{\partial T_1^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{1ij}^{\mathrm{Veff}}\frac{\partial T_1^0}{\partial x_i}\right) \quad (6.1)$$

$$\tilde{C}_2\frac{\partial T_2^0}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda_{2ij}^{\mathrm{Veff}}\frac{\partial T_2^0}{\partial x_i}\right) \quad (6.2)$$

with $\lambda_{1ij}^{\mathrm{Veff}} = \lambda_{1ij}^{\mathrm{IVeff}} = \lambda_{1ij}^{\mathrm{IIIeff}}$, $\lambda_{2ij}^{\mathrm{Veff}} = \lambda_{2ij}^{\mathrm{IVeff}} = \lambda_{2ij}^{\mathrm{IIIeff}}$.

For the first order of approximation the thermal fluxes in the two constituents are independent. There is a continuous passage from model IV to model V on removing the interfacial conductance h. On an other hand, it is easy to show that $Q_l = O(\varepsilon^p)$, $p > 3$, also leads to model V.

7. BILAMINATED COMPOSITE

Consider the particular geometry of a periodic layered medium (Fig. 3). This very simple geometry

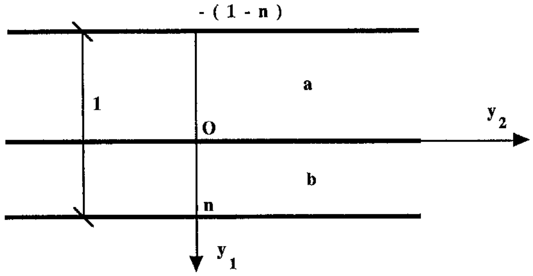

Fig. 3. Bilaminated composite.

is interesting to investigate because it permits analytical results, and because more realistic composites present similar general behaviour. The constituents are not connected in direction y_1. n denotes the volume fraction of component 2. l is the period in direction X_2, i.e. 1 is the period in direction y_2. Each constituent is assumed isotropic and homogeneous, with conductivities a and b for constituents 1 and 2, respectively. The medium exhibits a symmetry of revolution around the y_1-axis. Therefore in all cases we have

$$\lambda_{22}^{\mathrm{eff}} = \lambda_{33}^{\mathrm{eff}} \quad \lambda_{12}^{\mathrm{eff}} = \lambda_{13}^{\mathrm{eff}} = \lambda_{23}^{\mathrm{eff}} = 0.$$

The classical case, model I, has been already investigated in ref. [18]. The effective conductivity was shown to be given classically by

$$\lambda_{22}^{\mathrm{Ieff}} = (1-n)a+nb \quad \lambda_{11}^{\mathrm{Ieff}} = \frac{ab}{na+(1-n)b}.$$

Considering model II, it is easy to check that the conductivity in direction y_2 is unchanged:

$$\lambda_{22}^{\mathrm{IIeff}} = \lambda_{22}^{\mathrm{Ieff}} = (1-n)a+nb.$$

In direction y_1 one obtains

$$\lambda_{11}^{\mathrm{IIeff}} = \frac{1}{\dfrac{1}{\lambda_{11}^{\mathrm{Ieff}}}+\dfrac{2}{lh}}$$

which is similar to relation (13) in ref. [2].

Decreasing the thermal barrier resistance, i.e. increasing h, leads continuously to model I.

Model III shows the influence of the non-connection of the components. For the first order of approximation the composite is not a conductor in direction y_1. In the other directions, due to the particular geometry, and the homogeneity and isotropy of the components, the conductivity is unchanged:

$$\lambda_{11}^{\mathrm{IIIeff}} = 0$$

$$\lambda_{22}^{\mathrm{IIIeff}} = \lambda_{22}^{\mathrm{IIeff}} = \lambda_{22}^{\mathrm{Ieff}} = (1-n)a+nb.$$

When h becomes zero $\lambda_{ij}^{\mathrm{IIeff}}$ becomes $\lambda_{ij}^{\mathrm{IIIeff}}$ and at the same time model II gives model III. However, model III does not yield model II with increasing h.

Finally, models IV and V introduce the following effective parameters:

$$\lambda_{11}^{\mathrm{Veff}} = \lambda_{11}^{\mathrm{IVeff}} = 0$$

$$\lambda_{1ij}^{\mathrm{Veff}} = \lambda_{1ij}^{\mathrm{IVeff}} = (1-n)a$$

$$\lambda_{2ij}^{\mathrm{Veff}} = \lambda_{2ij}^{\mathrm{IVeff}} = nb \quad i = j = 2$$

$$H = \frac{2h}{l}.$$

Model V is obtained from model IV by putting $h = 0$. However, there is no continuous passage to model III.

8. CONCLUDING REMARKS

The macroscopic description of heat transfer in a composite with thermal barrier resistance on the interface of the components has been shown to strongly depend on the relative value of the barrier resistance with respect to the resistance of the components. Five very different macroscopic models are obtained whose domains of validity are related to the value of the dimensionless number $Q_l = (hl)/|a|$. They are shown in Fig. 2.

Following the analysis in Section 7, the five models can be regrouped into two classes. The first class comprises models I–III. The most powerful model is model II since it gives models I and III on increasing or decreasing h, respectively. The second class contains models IV and V. Here the most powerful model is model IV which yields model V on decreasing h. There is no continuous passage from one class to the other.

The value of Q_l has to be determined as a function of the power of the small parameter ε that measures the scale separation. Q_l is well defined from a knowledge of the micro structure of the composite. On the contrary, the value of ε requires the value of the characteristic macroscopic length L. This depends on the size of the sample or is related to the magnitude of the macroscopic gradient of the temperature since, roughly:

$$L = \mathrm{O}\left(T^0 \bigg/ \frac{\partial T^0}{\partial X}\right).$$

For example, in the case of a wavy temperature field of wave length λ, a good approximation of L is

$$L = \frac{\lambda}{2\pi}.$$

Changing L will change ε so that the value of Q_l with respect to the power of ε could be modified. Consequently, the correct model to describe the macroscopic behaviour would also be changed. Therefore, the macroscopic model of a given composite depends on the size of the macroscopic sample and on the excitation.

Finally it is worthwhile noticing that the macroscopic model for an n-constituent composite in cases IV and V is an n-temperature field model.

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